Recursive Partial Realization from the Combined Sequence of Markov Parameters and Moments

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ABSTRACT

In this paper, we develop the recursive structure of partial realization from a two-sided sequence of Markov parameters and moments. The specific feature of this recursive structure is that it accommodates sequences growing in two directions, like in the two-point Padé approximation. We present a recursive algorithm for the minimal partial realization from a sequence of Markov parameters and moments that amounts to recursive *LU* factorization of a Toeplitz behavior matrix, leaping over nongeneric sections. It turns out that this algorithm reveals a new continued-fraction expansion which extends the classical Cauer third form. As a corollary, various results related to Toeplitz matrices, two-point Padé approximations, continued fractions, etc. are unified within this partial-realization setting.

1. INTRODUCTION

Since Kalman formalized the partial realization from Markov parameters [1] in the system-theory context, many authors have investigated various aspects of the problems. It is intimately connected to such topics as the Berlekamp-Massey algorithm [6, 11] in the coding context, Magnus's principal-part continued-fraction expansion, the Euclidean algorithm, the factorization of Hankel matrices and Lanczos orthogonal polynomials [2–10, 38], the one-point Padé-approximation problem [7, 34–35], etc.

*This research was supported by NSF Grant DMS-870395.

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655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

In this paper, we rather investigate the partial realization from the combined sequence of Markov parameters and moments (CSMM). The Markov parameters h_k are defined to be the coefficients of the expansion of the transfer function around $z = \infty$, while the moments m_k are rather the coefficients of the expansion around $z = 0$, assuming the transfer function to be analytic at both 0 and ∞. To be more specific, our problem is to find a transfer function $h(z)$ of minimal degree that interpolates the data

$$s_n = [m_{n-1}, \dots, m_0, h_1, \dots, h_n].$$

As shown in our previous work [12], the key issue is the fact that the denominator of $h = b/a$ is given by a Toeplitz system of equations

$$aT = -c,$$

where T is the Toeplitz matrix constructed with s_n, and c is the shifted version of the bottom row of T.

Assume for a moment that the CSMM grows in one direction (i.e., either on the side of the h's or on the side of m's). Then, it is easily seen that the Toeplitz system of equations can be relabeled into a Hankel system of equations [33]. More concretely, the CSMM starting at one point and growing on one side is generated by a linear feedback shift register with connection polynomial $a(z)$. A connection polynomial for the one-sided sequence can be computed efficiently and recursively using the Berlekamp-Massey algorithm, which is nothing other than the LU factorization of the Hankel matrix [11]. All of this means that, in the situation of a CSMM growing in one direction, we are back to the classical partial-realization problem or the one-point Padé-approximation problem.

However, as shown in our previous work [12], the recursive algebraic structure of the CSMM can be grasped only in an incomplete and inaccurate fashion when the CSMM is restricted to grow in one direction. This is, in part, due to the arbitrariness of picking the starting point of the semiinfinite CSMM. Putting it another way, the difficulty is to assess the minimal interpolating function that is analytic at $z = 0$ [12, Example 4].

The purpose of this paper is therefore to develop the recursive structure of the partial realization from the CSMM when the data grow bidirectionally, that is, as $s_{n-1}, s_n, s_{n+1}, \dots$. The first issue is the behavior of the degree of the minimal partial realization of s_n as n increases. The degree of the minimal partial realization of s_n turns out to be the minimal rank of the two-sided extension of the partially defined Toeplitz matrix. This strongly contrasts with the one-sided extension of a partially defined Hankel matrix. The minimal-rank two-sided extension of the partially defined Toeplitz matrix is accomplished with the help of Iohvidov's theory [15]. Next, the recursive solution of $aT = -c$ is considered, and the existence of nongeneric sections as well as the Gohberg-Semencul formula [22] are reinterpreted in this particular partial-realization context. Finally, a generalized continued-fraction expansion that can accommodate nongeneric sections in T is developed.

Up to a certain extent, our problem is the two-point Padé approximation. Remember that in that approximation [28–29], given two formal expansions

$$L = c_0 + c_1 z + c_2 z^2 + \cdots,$$

$$L^* = c^*_{-1} z^{-1} + c^*_{-2} z^{-2} + \cdots$$

and a pair of integers (m, n), the problem is to find a rational fraction

$$\frac{P_{m,n}}{Q_{m,n}} = \frac{p_0 + p_1 z + \cdots + p_m z^m}{q_0 + q_1 z + \cdots + q_n z^n}$$

such that

$$Q_{m,n} L - P_{m,n} = O(z^r),$$

$$Q_{m,n} L^* - P_{m,n} = O(z^{-s})$$

where r and s are powers as large as possible. In this paper, however, we rather adopt the criterion

$$L - \frac{P_{m,n}}{Q_{m,n}} = O(z^r),$$

$$L^* - \frac{P_{m,n}}{Q_{m,n}} = O(z^{-s}).$$

As emphasized in [28], the two schemes are not equivalent, because in the former (twopoint Padé problem) $P_{m,n}/Q_{m,n}$ is not required to be analytic at 0 and ∞. In other words, in the twopoint Padé problem, L and L^* need not be the expansions around $z = 0$ and $z = \infty$ of the same function, and the coefficients of L and L^* need not be related [29]. On the other hand, in our problem, L and L^* are expansions around 0 and ∞ of a unique function, so that the coefficients of L and L^* are related through a linear feedback shift register, which is precisely the starting point of our approach.

Needless to say, the present paper involves such diverse topics as partial realization, fast LU factorization of Toeplitz matrices and Szegö biorthogonal polynomials, inversion of Toeplitz matrices with singular sections, continued fractions, etc. A deeper aim of the paper is to put together, within the specific context of the CSMM, a variety of results which have remained scattered over a voluminous literature.

An outline of the paper follows. In Section 2, we briefly review our previous work [12] and summarize the basic algebraic properties of the CSMM. In Section 3, the Iohvidov indexes are used to determine where and by how much the minimal degree "jumps." In Section 4, we present a recursive algorithm to update the connection polynomial. This algorithm, in Section 5, is connected with a new continued-fraction expansion for a transfer function which is analytic at both zero and infinity. Finally, in Section 6, the application to the inversion of Toeplitz matrices is also made.

2. COMBINED SEQUENCE OF MARKOV PARAMETERS AND MOMENTS

Consider a strictly proper rational function $h_d(z)$ which is analytic at $z=0$ and $z=\infty$. Therefore, we have two MacLaurin series expansions, one at $z=\infty$ and the other at $z=0$,

$$h_d(z) = \frac{b_1 z^{d-1} + b_2 z^{d-2} + \cdots + b_d}{z^d + a_1 z^{d-1} + \cdots + a_d} = \frac{b(z)}{a(z)}$$

$$= \begin{cases} h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + h_4 z^{-4} + \cdots, & |z| > R, \\ -m_0 - m_1 z - m_2 z^2 - m_3 z^3 - \cdots, & |z| < r, \end{cases} \tag{1}$$

where $R \geqslant r$ for the uniqueness of the Laurent series expansion. Note that $h_d(z)$ being analytic at $z=0$ means that $a_d \neq 0$.

Equating terms of equal powers of z in the series (1), it is easy to get the following equations:

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ a_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ a_{d-1} & a_{d-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_d \end{bmatrix} \tag{2}$$

$$= - \begin{bmatrix} m_0 & m_1 & \cdots & m_{d-1} \\ & m_0 & \ddots & m_{d-2} \\ & & \ddots & \vdots \\ & & & m_0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}. \tag{3}$$

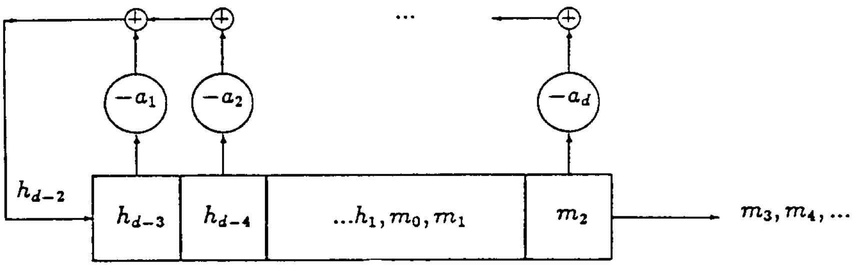


FIG. 1. LFSR for combined Markov parameters and moments.

The above would be the twopoint Padé equations if it were not for the extra requirement that $a_d \neq 0$. Rearranging these equations yields

LEMMA 1 [12]. *The combined sequence of Markov parameters and movements,* CSMM $= (\ldots, m_2, m_1, m_0, h_1, h_2, \ldots)$, *of* $h_d(z)$ *is generated on its left side by the linear feedback shift register* (LFSR), *with connection polynomial* $a(z)$, *as shown in Figure* 1. *A similar statement holds for the right side.*

In other words, (2) and (3) yield the compact form:

$$[a_d, \ldots, a_1, 1] \cdot \left[\begin{array}{cc|cccc|cc} \cdots & h_1 & m_0 & \cdots & & m_d & m_{d+1} & \cdots \\ \ddots & h_2 & h_1 & \ddots & & m_{d-1} & m_d & \ddots \\ \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots & \\ \cdots & h_{d+1} & h_d & \cdots & & m_0 & m_1 & \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc|cc} \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \end{array}\right]. \tag{4}$$

Denote by T_{d+1} the Toeplitz matrix inside the window of Equation (4). At this stage, the notion of bidirectionality (which is related to analyticity at $z = 0$, i.e., $a_d \neq 0$) is introduced. Then the Kronecker-like theorem is:

THEOREM 1 [12]. $h_d(z)$ *is irreducible with degree* d *and analytic at* $z = 0$ *if and only if the rank of* T_N *is* d *for all* $N \geq d$.

Only a *Toeplitz* test that incorporates the bidirectionality of the CSMM is able to predict the degree of the minimal realization analytic at $z = 0$. The Hankel test fails; see [12].

Now, consider the minimal partial realization for the index symmetric sequence of n Markov parameters and n moments, i.e., $s_n = [m_{n-1},\dots,m_0,h_1,\dots,h_{n-1},h_n]$. The issue of the minimal-degree realization from any combination of Markov parameters and moments has been solved in [12]. Our goal in this paper is to find a minimal realization algorithm in a nested manner when the data are increased in both directions, i.e., $s_1, s_2,\dots,s_n$.

To be more precise, from Theorem 1 we learn that the Toeplitz behavior matrix

$$T_\infty = \left\lceil \begin{array}{cccccc} m_0 & m_1 & \cdots & m_{n-1} & ? & ? \\ h_1 & m_0 & \ddots & m_{n-2} & m_{n-1} & ? \\ \vdots & & \ddots & & & \ddots \\ h_{n-1} & h_{n-2} & \cdots & m_0 & m_1 & \\ h_n & h_{n-1} & \ddots & h_1 & m_0 & \ddots \\ ? & h_n & & & \ddots & \\ ? & ? & \ddots & & & \ddots \end{array} \right. \tag{5}$$

should be extended in such a way that rank $T_K = d_n$ for all $K \geqslant d_n$ and d_n as small as possible. Namely, d_n can be characterized as the minimum rank of T_∞ over all extensions ? of s_n. It turns out that there is also a jump property for the minimal degree in the extension of (5). Accordingly, to have a nested realization structure, it is necessary to update the connection polynomials in a recursive manner. These are the topics of the next two sections.

3. DEGREE OF MINIMAL REALIZATION

For the sake of completeness, we first review the algebraic theory of the Iohvidov index and then formulate the minimal-degree realization theorem. For a more comprehensive discussion, the reader is referred to [12].

Consider the set of all successive principal submatrices $T_1,\dots,T_r,T_{r+1},\dots,T_n$. Let T_r be the largest nonsingular submatrix in this set. Thus, the matrix T_{r+1} is of rank r and nullity one. Following Theorem 1, there are two uniquely defined infinite sequences $h'_{r+1}, h'_{r+2},\dots,m'_{r+1}, m'_{r+2},\dots$ yielding the singular extension T'_∞ of the truncated matrix T_{r+1} through the recursion

$$h'_{r+k} + a'_1 h'_{r+k-1} + \cdots + a'_r h'_k = 0$$

and

$$m'_k + a'_1 m'_{k+1} + \cdots + a'_r m'_{r+k} = 0, \qquad k = 1,2,3,\ldots,$$

where the a'_i's are determined by $[a'_r, \ldots, a'_1, 1]T_{r+1} = 0$.

Define two integers, k and l, through the relations

$$\begin{aligned} &h_{r+1} = h'_{r+1}, \ldots, \quad h_{n-k-1} = h'_{n-k-1}; \qquad h_{n-k} \neq h'_{n-k} \\ &m_{r+1} = m'_{r+1}, \ldots, \quad m_{n-l-1} = m'_{n-l-1}; \qquad m_{n-l} \neq m'_{n-l}. \end{aligned} \tag{6}$$

In other words, the indexes (k, l) specify where the minimal-degree extension of T_{r+1} starts to differ from the original sequence. This can be better understood from the diagram

$$T_n = \left[\begin{array}{ccc|ccccc} m_0 & \cdots & m_{r-1} & \cdots & m_{n-l} & \cdots & m_{n-1} \\ \vdots & \ddots & & & & & \vdots \\ h_{r-1} & & m_0 & & & & \vdots \\ \hline \vdots & & & \cdot & & & m_{n-l} \\ h_{n-k} & & & & \cdot & & \vdots \\ \vdots & & & & & \cdot & \vdots \\ h_{n-1} & \cdots & \cdots & h_{n-k} & \cdots & \cdots & m_0 \end{array}\right].$$

Then we have Iohvidov's rank law as follows.

THEOREM 2 [15]. *If T_n is a Toeplitz matrix with (r, k, l) characteristic, then its rank is $\rho = r + k + l$.*

With the help of the Iohvidov index of Toeplitz matrices, the minimal degree d_n of the partial realization for s_n is the following.

THEOREM 3 [12]. *Given the data s_n, d_n is either ρ or $L - \rho$, where*

$$\rho = \operatorname{rank}\begin{bmatrix} & T_n & \\ h_n & \cdots & h_1 \end{bmatrix},$$

$$L = 2n + 1.$$

4. RECURSIVE MINIMAL-REALIZATION ALGORITHM

We now develop a recursive algorithm, based on the Iohvidov index, such that the partial realization can be solved not only in a bidirectional manner, but also leaping over singular sections. In a certain sense, the proposed algorithm is a bidirectional version of the Berlekamp-Massey algorithm, i.e., one Markov parameter and one moment are matched at every step of the algorithm. Moreover, this algorithm reveals a new continued-fraction expansion which extends the classical Cauer third form [24].

4.1. *Generic Case*

To begin with, let us first consider the problem of computing the connection polynomial $\mathbf{a}_{j+1}$ matching s_{j+1} from the connection polynomials $\mathbf{a}_j$ and $\mathbf{a}_{j-1}$. The key is the following LU step of the factorization of T_{n+1} where the moment m_n is considered to be an undetermined parameter:

$$\begin{bmatrix} z\mathbf{a}_{j-1} \\ \mathbf{a}_j \\ z\mathbf{a}_j \end{bmatrix} T_{n+1} = \begin{bmatrix} 0 & a_{j-1,j-1} & \cdots & a_{j-1,1} & 1 & 0 & \cdots & 0 \\ a_{j,j} & a_{j,j-1} & \cdots & a_{j,1} & 1 & 0 & \cdots & 0 \\ 0 & a_{j,j} & \cdots & a_{j,2} & a_{j,1} & 1 & \cdots & 0 \end{bmatrix} T_{n+1}$$

$$= \left[\begin{array}{ccccc|ccc} u_j^{(j-1)} & 0 & \cdots & 0 & u_0^{(j-1)} & u_{-1}^{(j-1)} & \cdots & u_{-n+j}^{(j-1)} \\ 0 & 0 & \cdots & 0 & u_0^{(j)} & u_{-1}^{(j)} & \cdots & u_{-n+j}^{(j)} \\ u_{j+1}^{(j)} & 0 & \cdots & 0 & 0 & u_0^{(j)} & \cdots & u_{-n+j+1}^{(j)} \end{array}\right], \tag{7}$$

where

$$\begin{aligned} u_{j+1}^{(j)} &= h_{j+1} + a_{j,1}h_j + \cdots + a_{j,j}h_1, \\ u_j^{(j-1)} &= h_j + a_{j-1,1}h_{j-1} + \cdots + a_{j-1,j-1}h_1, \\ u_0^{(j)} &= m_0 + a_{j,1}m_1 + \cdots + a_{j,j}m_j, \\ u_0^{(j-1)} &= m_0 + a_{j-1,1}m_1 + \cdots + a_{j-1,j-1}m_{j-1}, \end{aligned} \tag{8}$$

and z denotes the right shift operation.

Let us express $\mathbf{a}_{j+1}$ as

$$\mathbf{a}_{j+1} = (z + \alpha_j)\mathbf{a}_j + \beta_j z \mathbf{a}_{j-1}, \qquad j = 0, 1, \ldots, n-1, \tag{9}$$

through a three-term recursion and get

$$\mathbf{a}_{j+1}T_{n+1} = \left[0,\ldots,0 \mid u_0^{(j+1)},\ldots,u_{-n+j+1}^{(j+1)}\right] \tag{10}$$

with

$$\begin{aligned} \beta_j &= -u_{j+1}^{(j)}/u_j^{(j-1)}, \\ \alpha_j &= -u_0^{(j-1)}\beta_j/u_0^{(j)}, \end{aligned} \tag{11}$$

and the initial conditions

$$\begin{aligned} \mathbf{a}_{-1} &= [0,0,\ldots,0] \\ \mathbf{a}_0 &= [1,0,\ldots,0] \\ \beta_0 &= h_1 \\ \alpha_0 &= -h_1/m_0. \end{aligned} \tag{12}$$

From the point of view of minimal recursive realization, each $\mathbf{a}_j$ in the factorization (7)–(12) corresponds to the connection polynomial for matching the data s_j. One defines $u_j^{(j-1)}$ to be the forward residue associated with $\mathbf{a}_{j-1}$; this definition is motivated by the LFSR interpretation of (8). This residue vanishes in the degenerate case where the updated sequence s_j is generated by the updated connection polynomial $z\mathbf{a}_{j-1}$, which is not admissible because of its root at zero. On the other hand, one defines $u_0^{(j)}$ to be the backward residue of $\mathbf{a}_j$; $u_0^{(j)} = 0$ means that T_{j+1} is singular. The above factorization cannot proceed if some forward and/or backward residue vanishes. Thus, we define:

DEFINITION 1. A transfer function which is analytic at zero and infinity is *generic* if both forward and backward residues are all nonvanishing. Otherwise, it is *nongeneric*.

In addition, in view of (2) and (9), one can also derive the three-term recursion for the corresponding numerator vectors as follows:

$$\begin{aligned} \mathbf{b}_{j+1} &= \left[b_{j+1,j+1}, b_{j+1,j},\ldots,b_{j+1,1},0,\ldots,0\right] \\ &= (z+\alpha_j)\mathbf{b}_j + \beta_j z\mathbf{b}_{j-1}, \qquad j = 1,2,\ldots,n-1, \end{aligned} \tag{13}$$

with initial conditions

$$\begin{aligned} \mathbf{b}_0 &= [0,0,\ldots,0], \\ \mathbf{b}_1 &= [h_1,0,\ldots,0]. \end{aligned} \tag{14}$$

At this point, $\mathbf{a}_j$ and $\mathbf{b}_j$ determine the denominator and numerator polynomials, respectively, of the realization of s_j. This completes the recursive minimal-realization algorithm in the generic case.

Equations (7)–(12) provide a fast triangular factorization of T_{n+1}, because the number of multiplications is proportional to n^2. This factorization exists only in the generic case and is different from the classical two-term recursion Schur factorization [20, 21]. We note that Kung used the LU factorization of a Hankel matrix [10] for the minimal partial realization from Markov parameters, while here we use the LU factorization of a Toeplitz matrix for the minimal partial realization from symmetric CSMM. These two algorithms are not mapped into each other by the relabeling operation that maps a Toeplitz matrix into a Hankel matrix [33], because our algorithm is symmetric in the sense that one Markov parameter and one moment are added at a time. Specifically, the connection polynomials in the Hankel factorization are related to the Lanczos polynomials, while those in the Toeplitz factorization are rather related to the Szegö biorthogonal polynomials [20]. Moreover, we shall see in the nongeneric case that, in addition to the singular sections as in the Hankel factorization [10], we also have to handle the analyticity at zero. The generalized factorizations covering these two degenerate cases will be treated in the following subsections.

4.2. *Nongeneric Case*

In addition to generic sections (GS) in the above factorization, there are two types of sections in the nongeneric case: the vanishing forward residue sections (VFRS) corresponding to nonanalyticity at $z=0$, and the singular sections (SS) corresponding to vanishing backward residues. At this point, the above factorization has to be generalized to accommodate these nongeneric cases.

Suppose a vanishing forward residue arises in the above factorization, i.e., $\mathbf{a}_j T_{n+1} = [0,\ldots,0,u_0^{(j)},\ldots,u_{-n+j}^{(j)}]$ and $u_{j+1}^{(j)} = 0$. Moreover, the forward residues keep on vanishing until one nonvanishing residue appears at the tth step, i.e.,

$$z^i \mathbf{a}_j T_{n+1} = \left[0,\ldots,0,u_0^{(j)},\ldots,u_{-n+j+i}^{(j)}\right], \qquad i=0,1,\ldots,t-1,$$

and

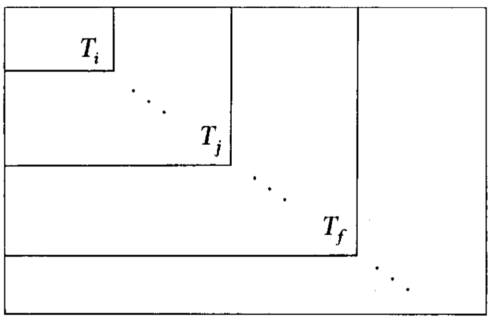
$$z^t \mathbf{a}_j T_{n+1} = \left[u_{j+t}^{(j)}, 0, \dots, 0, u_0^{(j)}, \dots, u_{-n+j+t}^{(j)} \right].$$

At this point, let us define t as follows.

DEFINITION 2. The *residue index* t of a connection vector is the extra size at which the first nonvanishing forward residue appears.

Consequently, we have to leap over these vanishing residue sections, since in them there are no "admissible" denominators to be analytic at zero. Therefore, assume T_i, T_j, and T_f are nonsingular, and let us consider the problem of computing $\mathbf{a}_f$ from $\mathbf{a}_i$ and $\mathbf{a}_j$ in a three term recursion:

$T_{n+1} =$



(15)

The nongeneric sections occur from T_{i+1} to T_{j-1} and/or from T_{j+1} to T_{f-1}. As will become clear later, the structure of the three-term recursion is determined by the nature of the section occurring between T_{j+1} and T_{f-1}; the sections between T_{i+1} and T_{j-1} (GS or VFRS or SS) only affect the amount of shifting on $\mathbf{a}_i$. Therefore, we have three relevant combinations (the sections between T_{i+1} and T_{j-1} are written before the slash, while the sections between T_{j+1} and T_{f-1} are written after the slash):

(GS, VFRS, SS)/GS,
(GS, VFRS, SS)/VFRS,
(GS, VFRS, SS)/SS.

4.2.1. (GS, VFRS, SS)/GS. GS/GS has already been discussed in connection with the factorization (7)–(12). In the case of VFRS/GS, it is easy to see that only the term $z\mathbf{a}_{j-1}$ in (7) needs to be replaced by $z^s\mathbf{a}_i$, where s denotes the residue index of $\mathbf{a}_i$. Similarly, in the case of SS/GS, it is replaced by $z^{l_1}\mathbf{a}_i$, where l_1 is in the Iohvidov index (i, k_1, l_1) of T_j. As a result, in the last two situations, $\alpha_{f,f} \neq 0$ is also true, which assures analyticity at zero.

4.2.2. (GS, VFRS, SS)/VFRS. Let us first consider VFRS/VFRS, i.e., we have two successive connection vectors, $\mathbf{a}_i$ and $\mathbf{a}_j$, with residue indexes s, t respectively in (15). We observe that

$$A_f T_{n+1} = \begin{bmatrix} z^s\mathbf{a}_i \\ \mathbf{a}_j \\ \vdots \\ z^t\mathbf{a}_j \end{bmatrix} T_{n+1}$$

$$= \left[\begin{array}{cccccccc|ccc} u_j^{(i)} & 0 & \cdots & 0 & u_0^{(i)} & \cdot & \cdot & \cdot & u_{-t+1}^{(i)} & u_{-t}^{(i)} & \cdots & u_{-n+j}^{(i)} \\ 0 & 0 & \cdots & 0 & u_0^{(j)} & \cdot & \cdot & \cdot & u_{-t+1}^{(j)} & u_{-t}^{(j)} & \cdots & u_{-n+j}^{(j)} \\ \vdots & & & & & & & & \vdots & \vdots & & \vdots \\ u_f^{(j)} & 0 & \cdots & 0 & 0 & \cdot & \cdot & \cdot & 0 & u_0^{(j)} & \cdots & u_{-n+f}^{(j)} \end{array}\right]. \tag{16}$$

Note that, by Definition 2, $j = i + s$ and $f = j + t$.

To construct $\mathbf{a}_f$, let us take the row operation on (16),

$$W_f = [w_1, w_2, \ldots, w_{t+1}, 1],$$

such that all the elements of the leftmost f columns of (16) are vanishing, i.e.,

$$\mathbf{a}_f T_{n+1} = W_f A_f T_{n+1} = \left[0,\ldots,0, \mid u_0^{(f)}, \ldots, u_{-n+f}^{(f)}\right]. \tag{17}$$

Equivalently, we need to solve

$$\begin{bmatrix} u_j^{(i)} & & & \\ u_0^{(i)} & u_0^{(j)} & & \\ \vdots & \vdots & \ddots & \\ u_{-t+1}^{(i)} & u_{-t+1}^{(j)} & \cdots & u_0^{(j)} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} -u_f^{(j)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{18}$$

which indeed has a unique solution because $u_j^{(i)}$ and $u_0^{(j)}$ are nonzero. Therefore, we get the three-term recursion as the following:

$$\mathbf{a}_f = \left(z^t + w_{t+1}z^{t-1} + \cdots + w_2\right)\mathbf{a}_j + w_1 z^s \mathbf{a}_i. \tag{19}$$

We remark that there is a fast inversion algorithm for a triangular Toeplitz matrix as in (18) [32].

In the case of GS/VFRS, it is easy to see that $s=1$ in $z^s\mathbf{a}_i$ of (16). The reader can also check for himself that if there are singular sections between T_{i+1} and T_{j-1}, i.e., SS/VFRS, only the term $z^s\mathbf{a}_i$ in (16) needs to be replaced by $z^{l_1}\mathbf{a}_i$, where l_1 is in the Iohvidov index (i, k_1, l_1) of T_j.

We remark that $a_{f,f} \neq 0$ in the solution of (17)–(18). This is easy to see, since both $u_j^{(i)}$ and $u_f^{(j)}$ being nonzero implies that $w_1 \neq 0$, which in turn implies that $a_{f,f} = a_{j,j}w_2 \neq 0$, because both $u_0^{(j)}$ and $u_0^{(i)}$ are nonzero.

Example 1. Let $s_5 = [1,0,1,1,2,3,\frac{9}{2},\frac{15}{4},-\frac{135}{8},-\frac{2769}{16}]$ be given. Then we have

$$T_6 = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & m_5 \\ 3 & 2 & 1 & 1 & 0 & 1 \\ \frac{9}{2} & 3 & 2 & 1 & 1 & 0 \\ \frac{15}{4} & \frac{9}{2} & 3 & 2 & 1 & 1 \\ -\frac{135}{8} & \frac{15}{4} & \frac{9}{2} & 3 & 2 & 1 \\ -\frac{2769}{16} & -\frac{135}{8} & \frac{15}{4} & \frac{9}{2} & 3 & 2 \end{bmatrix}$$

with m_5 an undetermined parameter. It is ready to see that $\mathbf{a}_1 = [-\frac{3}{2},1,0,0,0,0]$ has residue index 2, i.e.,

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & 1 & 0 & 0 \end{bmatrix} T_6$$

$$= \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & -\frac{3}{2} & 1-\frac{3}{2}m_5 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & -\frac{3}{2} \\ -3 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right].$$

From (18), we get $\mathbf{a}_3 = (z^2 - 6z - 4)\mathbf{a}_1 + z\mathbf{a}_0 = [6, 6, -\frac{15}{2}, 1, 0, 0]$ which also has the residue index 2. Thus, we get

$$\begin{bmatrix} 0 & 0 & -\frac{3}{2} & 1 & 0 & 0 \\ 6 & 6 & -\frac{15}{2} & 1 & 0 & 0 \\ 0 & 6 & 6 & -\frac{15}{2} & 1 & 0 \\ 0 & 0 & 6 & 6 & -\frac{15}{2} & 1 \end{bmatrix} T_6 = \left[\begin{array}{ccccc|c} -3 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 7+6m_5 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 3 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

and $\mathbf{a}_5 = (z^2 - 1)\mathbf{a}_3 + z^2\mathbf{a}_1 = [-6, -6, 12, 6, -\frac{15}{2}, 1]$.

4.2.3. (GS, VFRS, SS)/SS. In this subsection, we show how the above method carries over to the situation of the singular sections. Let us first consider SS/SS in (15). Namely, two successive singular sections take place from T_{i+1} to T_{j-1} and from T_{j+1} to T_{f-1}. Let the matrix T_j be of characteristic (i, k_1, l_1), and let T_f be of characteristic (j, k_2, l_2). Then $j = i + k_1 + l_1$ and $f = j + k_2 + l_2$.

Similar to the case of vanishing residue sections, we can also construct $\mathbf{a}_f$ through a three-term recursion. Moreover, this algorithm has a direct connection with the inversion of Toeplitz matrices with singular sections.

To this end, we observe the following:

$$A_f T_{n+1} = \begin{bmatrix} z^{k_2+l_1}\mathbf{a}_i \\ \mathbf{a}_j \\ \vdots \\ z^{l_2-1}\mathbf{a}_j \\ \hline z^{l_2}\mathbf{a}_j \\ \vdots \\ z^{l_2+k_2}\mathbf{a}_j \end{bmatrix} T_{n+1}$$

$$= \left[\begin{array}{ccc|ccc|ccc|c} u^{(i)}_{l_1+k_2+i} & \cdots & u^{(i)}_{l_1+i} & 0 & \cdots & 0 & u^{(i)}_{-k_1} & \cdots & u^{(i)}_{-(l_2+k_1-1)} & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & u^{(j)}_{-k_2} & \cdots & u^{(j)}_{-(k_2+l_2-1)} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & u^{(j)}_{-k_2} & \cdots \\ \hline u^{(j)}_{l_2+j} & & & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ u^{(j)}_{f} & \cdots & u^{(j)}_{l_2+j} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \end{array}\right]. \tag{20}$$

To construct $\mathbf{a}_f$, let us apply the row operation

$$Q_f = \left[q_1, q_{k_2+2}, \dots, q_{k_2+l_2+1} \,|\, q_{k_2+1}, \dots, q_2, 1\right]$$

such that all the elements of the leftmost f columns of (20) are vanishing:

$$\mathbf{a}_f T_{n+1} = Q_f A_f T_{n+1} = \left[0, \dots, 0 \,|\, u_0^{(f)}, \dots, u_{-n+f}^{(f)}\right]. \tag{21}$$

Equivalently, we need to solve the following system of equations:

$$\left[\begin{array}{cccc|ccc}
u^{(i)}_{l_1+i} & & & & & & \\
u^{(i)}_{l_1+i+1} & u^{(j)}_{l_2+j} & & & & & \\
\vdots & \vdots & \ddots & & & & \\
u^{(i)}_{l_1+i+k_2} & u^{(j)}_{f-1} & \cdots & u^{(j)}_{l_2+j} & & & \\
\hline
u^{(i)}_{-k_1} & 0 & \cdots & 0 & u^{(j)}_{-k_2} & & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \\
u^{(i)}_{-(k_1+l_2-1)} & 0 & \cdots & 0 & u^{(j)}_{-(k_2+l_2-1)} & \cdots & u^{(j)}_{-k_2}
\end{array}\right]
\left[\begin{array}{c}
q_1 \\ q_2 \\ \vdots \\ q_{k_2}+1 \\ \hline q_{k_2}+2 \\ \vdots \\ q_{k_2+l_2+1}
\end{array}\right]$$

$$= -\left[\begin{array}{c}
u^{(j)}_{l_2+j} \\ u^{(j)}_{l_2+j+1} \\ \vdots \\ u^{(j)}_{f} \\ \hline 0 \\ \vdots \\ 0
\end{array}\right]. \tag{22}$$

Then we get the three-term recursion

$$\mathbf{a}_f = \left(z^{l_2+k_2} + q_2 z^{l_2+k_2-1} + \cdots + q_{k_2+1} z^{l_2} + q_{k_2+l_2+1} z^{l_2-1} + \cdots + q_{k_2+2}\right)\mathbf{a}_j$$

$$+ q_1 z^{k_2+l_1}\mathbf{a}_i. \tag{23}$$

In the case of GS/SS and VFRS/SS, only the term $z^{k_2+l_1}\mathbf{a}_i$ in (20) is replaced by $z^{k_2+1}\mathbf{a}_i$ and $z^{k_2+s}\mathbf{a}_i$, respectively, where s is the residue index of $\mathbf{a}_i$ $(j = s+i)$. Note that $a_{f,f} \neq 0$ in the solution $\mathbf{a}_f$.

EXAMPLE 2. The following example from Delsarte et al. [17] can further illustrate our algorithm. We are given

$$T_8 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 3 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 2 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Since T_4 is the first nonsingular submatrix with characteristic (0,2,2), from (20) we have

$$\begin{bmatrix} \mathbf{a}_0 \\ z\mathbf{a}_0 \\ z^2\mathbf{a}_0 \\ z^3\mathbf{a}_0 \\ z^4\mathbf{a}_0 \end{bmatrix} T_8 = \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 1 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right],$$

and we get $\mathbf{a}_4 = (z^4 - z^3 + z - 1)\mathbf{a}_0 = [-1,1,0,-1,1,0,0,0]$. Since the next nonsingular submatrix T_6 is of characteristic (4,1,1), we have

$$\begin{bmatrix} z^3\mathbf{a}_0 \\ \mathbf{a}_4 \\ z\mathbf{a}_4 \\ z^2\mathbf{a}_4 \end{bmatrix} T_8 = \left[\begin{array}{cccccc|cc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & -2 \end{array}\right].$$

Again, taking row operations, we get $\mathbf{a}_6 = (z^2 + \frac{1}{2}z - 1)\mathbf{a}_4 - 2z^3\mathbf{a}_0$ and

$\mathbf{a}_6 T_8 = [1, -\frac{3}{2}, -\frac{1}{2}, 0, -\frac{3}{2}, -\frac{1}{2}, 1, 0] T_8 = [0,0,0,0,0,0, -2, -\frac{9}{2}]$. Now, T_7 can be viewed as of characteristic $(6,0,1)$. Thus, we have

$$\begin{bmatrix} z\mathbf{a}_4 \\ \mathbf{a}_6 \\ z\mathbf{a}_6 \end{bmatrix} T_8 = \left[\begin{array}{ccccccc|c} 2 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -\frac{9}{2} \\ -\frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

and get $\mathbf{a}_7 = (z - \frac{7}{4})\mathbf{a}_6 + \frac{7}{4} z \mathbf{a}_4 = [-\frac{7}{4}, \frac{15}{8}, \frac{9}{8}, -\frac{1}{2}, \frac{7}{8}, \frac{9}{8}, -\frac{9}{4}, 1]$ and $\mathbf{a}_7 T_8 = [0,0,0,0,0,0,0, \frac{33}{8}]$.

5. EXTENDED CAUER THIRD FORM

The existing continued-fraction expansions for matching the mixed Markov parameters and moments are the Cauer third form [24], the modified Cauer form [25] and the Perron-Carathéodory fraction [26]. They are valid, however, only for generic transfer functions. Here, the proposed recursive algorithm can be viewed as a generalized continued-fraction approach based on the generalized Toeplitz factorization technique in order to accommodate nongeneric sections for which the existing methods fail to be applicable.

Let us express the three-term recursion (9), (19), (23), (13) in polynomial notation:

$$\begin{aligned} a_{t_{i+1}}(z) &= q_{t_i}(z) a_{t_i}(z) + p_{t_i}(z) a_{t_{i-1}}(z), \\ b_{t_{i+1}}(z) &= q_{t_i}(z) b_{t_i}(z) + p_{t_i}(z) b_{t_{i-1}}(z), \qquad i = 1,2,\ldots, \end{aligned} \tag{24}$$

with initial conditions

$$a_{t_0}(z) = 1, \qquad a_{t_1}(z) = q_{t_0}(z), \qquad b_{t_0}(z) = 0, \qquad b_{t_1}(z) = p_{t_0}(z),$$

where

$$a_{t_i}(z) = z^{t_i} + a_{t_i,1} z^{t_i - 1} + \cdots + a_{t_i,t_i}$$

and

$$b_{t_i}(z) = b_{t_i,1}z^{t_i-1} + b_{t_i,2}z^{t_i-2} + \cdots + b_{t_i,t_i}.$$

Note that $p_{t_0}(z)$ is determined by $q_{t_0}(z)$ using (2) or (3). Then, we have

LEMMA 2. *Each approximant $b_{t_i}(z)/a_{t_i}(z)$ of order t_i can match t_i Markov parameters and t_i moments and is irreducible.*

Proof. The first part follows from the factorization algorithm constructed in the previous sections. We need to show coprimeness.

By a well-known technique [27], from (24) we can deduce

$$\begin{bmatrix} b_{t_{i-1}}(z) & b_{t_i}(z) \\ a_{t_{i-1}}(z) & a_{t_i}(z) \end{bmatrix} = \begin{bmatrix} 0 & p_{t_0}(z) \\ 1 & q_{t_0}(z) \end{bmatrix} \cdots \begin{bmatrix} 0 & p_{t_{i-1}}(z) \\ 1 & q_{t_{i-1}}(z) \end{bmatrix}. \tag{25}$$

Since $p_{t_v}(z)$, $v = 0, \dots, i-1$, are powers of z, we have

$$a_{t_i}(z)b_{t_{i-1}}(z) - a_{t_{i-1}}(z)b_{t_i}(z) = \prod_{v=0}^{i-1} p_{t_v}(z) = c_1 z^{c_2},$$

where c_1 is a constant and c_2 is a nonnegative integer. This implies that the greatest common divisor of $a_{t_i}(z)$ and $b_{t_i}(z)$ is a divisor of z^{c_2}. But $a_{t_i}(z)$ has no roots at zero, which in turn implies coprimeness of $a_{t_i}(z)$ and $b_{t_i}(z)$. ■

Moreover, it is easy to see from the theory of continued-fraction expansion [23, 27] that (25) indeed implies a continued fraction expansion of the form

$$h_d(z) = \cfrac{\ddots}{\ddots + \cfrac{p_{t_i}(z)}{q_{t_i}(z) + \cfrac{\ddots}{\ddots}}}, \tag{26}$$

where $p_{t_i}(z)$, $q_{t_i}(z)$ are of the form:

$$\begin{aligned} p_{t_i}(z) &= p_{t_i} z^{n_{t_i}}, \\ q_{t_i}(z) &= z^{m_{t_i}} + q_1 z^{m_{t_i}-1} + \cdots + q_{m_{t_i}}, \end{aligned} \tag{27}$$

and n_{t_i} and m_{t_i} are determined by the Iohvidov index or the residue index associated with the nongeneric sections. In particular, in the generic case, we have

$$h_d(z) = \cfrac{\beta_0}{(z+\alpha_0) + \cfrac{\beta_1 z}{(z+\alpha_1) + \ddots + \cfrac{\beta_{d-1} z}{z + \alpha_{d-1}}}} \tag{28}$$

which is exactly the Cauer third form [24]. Hence, we can view (26) as the extended Cauer third form.

Example 3. The transfer function of Example 1 can be written as

$$h_5(z) = \cfrac{3}{\left(z - \frac{3}{2}\right) + \cfrac{z}{\left(z^2 - 6z - 4\right) + \cfrac{z^2}{z^2 - 1}}},$$

which consists of one generic section and two vanishing forward residue sections. The transfer function of Example 2 can be written as

$$h_7(z) = \cfrac{z^2}{\left(z^4 - z^3 + z - 1\right) + \cfrac{-2z^3}{\left(z^2 + \frac{1}{2}z - 1\right) + \cfrac{\frac{7}{4}z}{z - \frac{7}{4}}}},$$

which consists of two singular sections and one generic section. Note that $p_{t_0}(z) = z^2$ is computed from (2) or (3).

6. INVERSION OF TOEPLITZ MATRICES WITH SINGULAR SECTIONS

Inversion of Toeplitz matrices with singular sections has been considered by several authors ([7, 16–18, 36–37], to mention but a few). In this section, we rather show how our algorithm can be extended in such a way as to handle this Toeplitz inversion problem.

Consider the problem of inverting T_f in (15). In order to apply the celebrated Gohberg-Semencul formula [22], we need the connection polynomials $\mathbf{a}_f$ and $\bar{\mathbf{a}}_f$ such that $\bar{\mathbf{a}}_f T_{n+1} = [1, \bar{a}_{f,1}, \dots, \bar{a}_{f,f}, 0, \dots, 0] T_{n+1} = [v_0^{(f)}, 0, \dots, 0, v_{-f-1}^{(j)}, \dots]$. We already know how to compute $\mathbf{a}_f$ from $\mathbf{a}_i$ and $\mathbf{a}_j$. All that remains is a recursive formula for $\bar{\mathbf{a}}_f$. Consider

$$\bar{A}_f T_{n+1} = \left[\begin{array}{c} z^{l_1+k_2}\bar{\mathbf{a}}_i \\ \bar{\mathbf{a}}_j \\ \vdots \\ z^{l_2}\bar{\mathbf{a}}_j \\ \hline z^{l_2+1}\bar{\mathbf{a}}_j \\ \vdots \\ z^{l_2+k_2}\bar{\mathbf{a}}_j \end{array}\right] T_{n+1}$$

$$= \left[\begin{array}{c|ccc|ccc|ccc|c} v_{k_2+l_1}^{(i)} & v_{k_2+l_1-1}^{(i)} & \cdots & v_{l_1}^{(i)} & 0 & \cdots & 0 & v_{-(i+k_1)}^{(i)} & \cdots & v_{-(i+k_1+l_2)}^{(i)} & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & v_{-(j+k_2)}^{(j)} & \cdots & v_{-f}^{(j)} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ v_{l_2}^{(j)} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & v_{-(j+k_2)}^{(j)} & \cdots \\ \hline v_{l_2+1}^{(j)} & v_{l_2}^{(j)} & & & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & & \vdots & & \vdots & \vdots & \ddots & \vdots & \\ v_{l_2+k_2}^{(j)} & v_{l_2+k_2-1}^{(j)} & \cdots & v_{l_2}^{(j)} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \end{array}\right]. \tag{29}$$

To construct $\bar{\mathbf{a}}_f$, let us apply the row operation

$$P_f = \left[p_1, 1, p_2, \dots, p_{l_2+1}, p_{l_2+k_2+1}, \dots, p_{l_2+2}\right]$$

on both sides of (29) in such a way that all the elements of the f columns starting from the 2nd and extending to the $(f+1)$th are vanishing. Namely,

$$\bar{\mathbf{a}}_f T_{n+1} = P_f \bar{A}_f T_{n+1} = \left[v_0^{(f)}, 0, \dots, 0, v_{f+1}^{(f)}, \dots\right]. \tag{30}$$

Equivalently, we need to solve the following system of equations:

$$
\left[\begin{array}{cccc|ccc}
v^{(i)}_{-(i+k_1)} & & & & & & \\
v^{(i)}_{-(i+k_1+1)} & v^{(j)}_{-(j+k_2)} & & & & & \\
\vdots & \ddots & \ddots & & & & \\
v^{(i)}_{-(i+k_1+l_2)} & v^{(j)}_{-(f-1)} & \cdots & v^{(j)}_{-(j+k_2)} & & & \\
\hline
v^{(i)}_{l_1} & 0 & \cdots & 0 & v^{(j)}_{l_2} & & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \\
v^{(i)}_{k_2+l_1-1} & 0 & \cdots & 0 & v^{(j)}_{l_2+k_2-1} & \cdots & v^{(j)}_{l_2}
\end{array}\right]
\left[\begin{array}{c}
p_1 \\ p_2 \\ \vdots \\ p_{l_2+1} \\ \hline p_{l_2+2} \\ \vdots \\ p_{l_2+k_2+1}
\end{array}\right]
$$

$$
= -\left[\begin{array}{c}
v^{(j)}_{-(j+k_2)} \\ v^{(j)}_{-(j+k_2+l_2)} \\ \vdots \\ v^{(j)}_{-f} \\ \hline 0 \\ \vdots \\ 0
\end{array}\right]. \tag{31}
$$

Note that $v^{(j)}_{-(j+k_2)}$, $v^{(i)}_{-(i+k_1)}$ and $v^{(j)}_{l_2}$ are all nonzero by definition of the Iohvidov indexes. Thus, Equation (31) has a unique solution. As noted before, if there are no singular sections between T_i and T_j, the term $z^{l_1+k_2}\bar{\mathbf{a}}_i$ in (29) should be replaced by $z^{k_2}\bar{\mathbf{a}}_i$, since for T_j the index $l_1 = 0$.

At this point, we can use the Gohberg-Semencul formula [22, 20] for the inversion of Toeplitz matrices with singular sections. An example of the inversion of a nonsymmetric Toeplitz matrix with singular sections is provided in the following.

EXAMPLE 4. We are given

$$
T_6 = \begin{bmatrix}
2 & 1 & 3 & 4 & 2 & 1 \\
-1 & 2 & 1 & 3 & 4 & 2 \\
3 & -1 & 2 & 1 & 3 & 4 \\
5 & 3 & -1 & 2 & 1 & 3 \\
0 & 5 & 3 & -1 & 2 & 1 \\
1 & 0 & 5 & 3 & -1 & 2
\end{bmatrix},
$$

where T_5 is of characteristic $(2,2,1)$. From (29), we have

$$\left[\begin{array}{cccccc} 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right] T_6$$

$$= \left[\begin{array}{c|cc|c|cc} \frac{1}{2} & -\frac{5}{2} & \frac{5}{2} & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & -5 & -5 \\ -9 & 0 & 0 & 0 & 0 & -5 \\ \hline -2 & -9 & 0 & 0 & 0 & 0 \\ 4 & -2 & -9 & 0 & 0 & 0 \end{array}\right].$$

Therefore, we need to solve

$$\left[\begin{array}{cc|cc} \frac{5}{2} & & & \\ \frac{5}{2} & -5 & & \\ \hline \frac{5}{2} & & -9 & \\ -\frac{5}{2} & & -2 & -9 \end{array}\right] \left[\begin{array}{c} p_1 \\ p_2 \\ \hline p_3 \\ p_4 \end{array}\right] = - \left[\begin{array}{c} -5 \\ -5 \\ \hline 0 \\ 0 \end{array}\right]$$

and get

$$\bar{\mathbf{a}}_5 T_6 = \left[1, -1, \tfrac{26}{81}, \tfrac{19}{81}, \tfrac{10}{81}, -\tfrac{5}{9}\right] T_6 = \left[\tfrac{371}{81}, 0, 0, 0, 0, 0\right].$$

On the other hand, to get $\mathbf{a}_5$, we have from (20)

$$\left[\begin{array}{cccccc} 0 & 0 & 0 & \frac{1}{2} & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] T_6 = \left[\begin{array}{ccc|c|c|c} \frac{5}{2} & \frac{13}{2} & \frac{5}{2} & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 5 & 5 \\ \hline 9 & 0 & 0 & 0 & 0 & 5 \\ 2 & 9 & 0 & 0 & 0 & 0 \\ -4 & 2 & 9 & 0 & 0 & 0 \end{array}\right].$$

Thus, we have to solve

$$\left[\begin{array}{ccc|c} \frac{5}{2} & & & \\ \frac{13}{2} & 9 & & \\ \frac{5}{2} & 2 & 9 & \\ \hline \frac{5}{2} & & & 5 \end{array}\right]\left[\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ \hline q_4 \end{array}\right] = -\left[\begin{array}{c} 9 \\ 2 \\ -4 \\ \hline 0 \end{array}\right]$$

and get

$$\mathbf{a}_5 T_6 = \left[-\tfrac{9}{5}, \tfrac{358}{405}, \tfrac{137}{405}, \tfrac{40}{81}, -\tfrac{2}{9}, 1\right] T_6 = \left[0,0,0,0,0, \tfrac{371}{81}\right].$$

With $\mathbf{a}_5$ and $\bar{\mathbf{a}}_5$, we can use the Gohberg-Semencul formula to compute the inverses of T_5 and T_6.

7. CONCLUSION

In this paper, we have presented a recursive algorithm for the minimal partial realization of a two-sided sequence of Markov parameters and moments. It turns out that this algorithm not only reveals a new type of

TABLE 1

	SM	CSMM
Structure	Hankel One-sided LFSR	Toeplitz Two-sided LFSR
Jump size	Iohvidov index	Iohvidov index
Connection polynomials	Lanczos	Szegö
Continued fraction	Magnus's principal part	Extended Cauer third form
Applications	System synthesis Geometric Control Cauchy index, stability Canonical Form	Identification and control from low and high frequency, etc.

continued-fraction expansion, analogous to Magnus's principal-part fraction of Hankel matrices, but also provides a method for the inversion of Toeplitz matrices with singular sections using the Gohberg-Semencul formula.

In summary, we have the diagram in Table 1, which compares our results with those of the conventional partial realization from the sequence of Markov parameters (SM).

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Received 12 May 1988; final manuscript accepted 29 November 1988